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On linear extension operators from growths of compactifications of products

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Abstract

We obtain some results on product spaces. Among them we prove that for noncompact spaces X_1 and X_2, the norm of every linear extension operator from $C(\beta(X_1 \times X_2) \setminus (X_1 \times X_2))$ into $C(\beta(X_1 \times X_2))$ is greater or equal than 2, and also that $\beta(X_1 \times X_2) \setminus (X_1 \times X_2)$ is not a neighborhood retract of $\beta(X_1 \times X_2)$.

Keywords: Product space; Stone–Čech compactification; Linear extension operator; Pseudocompact space

AMS classification: 54B10; 54C35; 54D35; 54C15

Introduction

In this paper all spaces considered are locally compact Hausdorff spaces. We write βX for the Stone–Čech compactification of the space X and X^* for the remainder $\beta X \setminus X$. In [1] it is proved that if X_1 and X_2 are noncompact spaces, then $(X_1 \times X_2)^*$ is not a retract of $\beta(X_1 \times X_2)$. By the Yoshizawa–Semadeni theorem [7] this result is equivalent to the nonexistence of multiplicative linear extension operators from $C((X_1 \times X_2)^*)$ to $C(\beta(X_1 \times X_2))$.

On the other hand, in [2] an example is given of spaces X_1 and X_2 such that there is no regular extension operator from $C((X_1 \times X_2)^*)$ into $C(\beta(X_1 \times X_2))$.

According with these results and since the norm of every linear extension operator is greater than or equal to 1, it seems natural to ask whether there can be a norm one linear extension operator from $C((X_1 \times X_2)^*)$ into $C(\beta(X_1 \times X_2))$. We prove that the norm of such an operator must be always greater than or equal to 2. Although one can think that this result relies heavily on the properties of Stone–Čech compactification, we obtain that

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this holds for every compactification K of the product of the noncompact spaces X_1 and X_2 such that $K \geqslant \widehat{X} \times \widehat{Y}$, provided that at least one of them is pseudocompact (here $\widehat{X}$ denotes the Alexandroff compactification of the space X). Also as an application of our results, we give an example in which the bound 2 is attained.

This paper is divided in four sections. The first one contains some preliminary facts needed in the sequel, in Section 2 the main technical result (Theorem 2.1) is stated and its consequences are derived from it, Section 3 contains the proof of Theorem 2.1 with some auxiliary lemmas and, finally, in Section 4 some examples and further remarks about the preceding results are given.

1. Preliminaries

We write $C(X)$ for the algebra of all real valued continuous functions on a space X. The space $C^*(X)$ (respectively $C_0(X)$) will be the subspace of $C(X)$ of all bounded functions (respectively of all functions vanishing at infinity). We write $\operatorname{coz} f = \{x \in X \mid f(x) \neq 0\}$, $Z(f) = f^{-1}(\{0\})$, and $\operatorname{supp} f = \mathrm{cl}_X \operatorname{coz} f$. If $0 \leqslant f \leqslant 1$ is a function of $C(X)$ we also define the set $Pf = \{x \in X \mid 0 < f(x) < 1\}$.

For a compact space K, let $M(K)$ denote the Banach space of all finite regular Borel measures on K with the norm $\|\mu\| = |\mu|(K)$, where $|\mu|$ is the total variation of μ. Write μ^+ and μ^- for the positive and the negative parts, respectively, of the measure μ. By the Riesz representation theorem, we identify the space $M(K)$ with $C(K)^*$, the dual space of $C(K)$.

We consider $M(K)$ equipped with the weak–star topology. Hence, a net (μ_i) in $M(K)$ converges to μ (in symbols $\mu_i \longrightarrow \mu$) if and only if $\lim \int f \,\mathrm{d}\mu_i = \int f \,\mathrm{d}\mu$ for each $f \in C(K)$.

A neighborhood base for $\mu \in M(K)$ is given by all the sets of the form

$$W(\mu, \{f_1, \ldots, f_n\}, \varepsilon) = \left\{\lambda \in M(K) \mid \left|\int f_i \mathrm{d}(\mu - \lambda)\right| < \varepsilon,\ i = 1, \ldots, n\right\},$$

where $\varepsilon > 0$ and $f_1, \ldots, f_n \in C(K)$.

For each $p \in K$, δ_p will be the Dirac measure on K with support $\{p\}$.

A topological space X is said *pseudocompact* if $C(X) = C^*(X)$. In [5], Glicksberg proves that a (Hausdorff completely regular) topological space is pseudocompact if and only if every infinite sequence of nonempty open sets has a cluster point (i.e., a point such that each of its neighborhoods meets infinitely many elements of the sequence).

2. The main theorem and some consequences

All the results of this paper are consequences of the following theorem, whose proof will be given in Section 3.

Theorem 2.1. *Let $\widehat{X}$ and $\widehat{Y}$ be the Alexandroff compactifications of the noncompact spaces X and Y respectively. Let $S = (\widehat{X} \times \widehat{Y}) \backslash (X \times Y)$. If X is pseudocompact and T is a linear extension operator from $C(S)$ in $C(\widehat{X} \times \widehat{Y})$, then $\|T\| \geqslant 2$.*

We need two simple lemmas about compactifications and linear extension operators. A *compactification* of a space X is a compact Hausdorff space K such that there exists an embedding ϕ of X as a dense subset of K. We do not distinguish notationally between X and $\phi(X)$. If K and K' are compactifications of X we write $K \leqslant K'$ if there exists a continuous map from K' onto K whose restriction to X is the identity. It is well known that $\beta X \geqslant K \geqslant \widehat{X}$ for any compactification K of X [4, p. 169]. The following fact is easily checked.

Lemma 2.2. *Let K' be a compactification of a space X and suppose that there exists a linear extension operator T' from $C(K' \backslash X)$ to $C(K')$. Then if K is a compactification of X such that $K \leqslant K'$, there is a linear extension operator T from $C(K \backslash X)$ to $C(K)$ such that $\|T\| \leqslant \|T'\|$.*

Lemma 2.3. *If K is a compactification of a space X and $K \backslash X$ is a neighborhood retract of K, then there exists a norm one linear extension operator from $C(K \backslash X)$ into $C(K)$.*

Proof. Suppose that r is a retraction from a neighborhood V of $K \backslash X$ onto $K \backslash X$. Let $h \in C(K)$ be such that $h(K \backslash \operatorname{int} V) = 0$ and $h(K \backslash X) = 1$. For each $g \in C(K \backslash X)$ let $\Omega(g)$ be the function on K given by

$$\Omega(g)(x) = h(x) g\big(r(x)\big) \quad \text{if } x \in \operatorname{int} V \text{ and } \Omega(g)(K \backslash \operatorname{int} V) = 0.$$

It is easily checked that Ω is the desired norm one linear extension operator. □

Corollary 2.4. *Let K be a compactification of the product $X \times Y$ of the noncompact spaces X and Y such that $K \geqslant \widehat{X} \times \widehat{Y}$, and assume that X is pseudocompact. Then*

(a) *Every linear extension operator from $C(K \backslash (X \times Y))$ to $C(K)$ has norm greater or equal than 2.*

(b) *The remainder $K \backslash (X \times Y)$ is not a neighborhood retract of K.*

Proof. Part (a) is a consequence of Theorem 2.1 and Lemma 2.2, since $K \geqslant \widehat{X} \times \widehat{Y}$.

Part (b) follows from Lemma 2.3. □

Theorem 2.5. *Let X and Y be noncompact spaces. Then:*

(a) *Every linear extension operator from $C((X \times Y)^*)$ to $C(\beta(X \times Y))$ has norm greater or equal than 2.*

(b) *The remainder $(X \times Y)^*$ is not a neighborhood retract of $\beta(X \times Y)$.*

Proof. (a) Suppose that there exists a linear extension operator from $C((X \times Y)^*)$ to $C(\beta(X \times Y))$. In this case it is easy to show that there exists a projection from

$C^*(X \times Y)$ onto $C_0(X \times Y)$ and according with a result of Conway [3], the space $X \times Y$ is pseudocompact. The space X being a continuous image of $X \times Y$ is also pseudocompact. Since $\beta(X \times Y) \geqslant \beta X \times \beta Y \geqslant \widehat{X} \times \widehat{Y}$, the result follows from Lemma 2.2 and Corollary 2.4.

(b) is a consequence of Lemma 2.3 and part (a). □

3. Proof of the main theorem

The underlying idea of the proof is a topological argument used in [1] which we adapt to the present context by using a certain functional analysis technique that appears in [2].

We prove first two lemmas.

Lemma 3.1. *Let $0 \leqslant f \leqslant 1$ be a continuous function on the compact space K and let p be a point such that $f(p) = 1$. Let $h \in C(K)$, $0 \leqslant h \leqslant 1$ and $\mu \in W(\delta_p, \{f, 1 - f\}, \varepsilon)$ such that $\|\mu\| \leqslant 1 + \eta$. Then:*

(1) *if $h(\operatorname{coz} f) = 1$ then $\int h \, \mathrm{d}\mu > 1 - 2\varepsilon - \eta/2$.*

(2) *if $h(Z(1 - f)) = 1$ and $|\mu|(Pf) < \varepsilon$ then $\int h \, \mathrm{d}\mu > 1 - 3\varepsilon - \eta/2$.*

(3) *if $h(\operatorname{coz} f) = 0$ then $\int h \, \mathrm{d}\mu < (\eta + 2\varepsilon)/2$.*

Proof. From the hypotheses it follows that $|\int f \, \mathrm{d}\mu - 1| < \varepsilon$ and $|\int (1 - f) \, \mathrm{d}\mu| < \varepsilon$, hence $1 + \eta \geqslant \int f \, \mathrm{d}|\mu| + \int (1 - f) \, \mathrm{d}|\mu| \geqslant |\int f \, \mathrm{d}\mu| + \int (1 - f) \, \mathrm{d}|\mu| > 1 - \varepsilon + \int (1 - f) \, \mathrm{d}|\mu|$ and so $\int (1 - f) \, \mathrm{d}|\mu| < \eta + \varepsilon$.

Since $|\int (1 - f) \, \mathrm{d}\mu| < \varepsilon$ we have also that $\int (1 - f) \, \mathrm{d}\mu^+ < (\eta + 2\varepsilon)/2$ and $\int (1 - f) \, \mathrm{d}\mu^- < (\eta + 2\varepsilon)/2$.

Now we prove (1). Since $h(\operatorname{coz} f) = 1$, we have $hf = f$, and therefore $\int h \, \mathrm{d}\mu = \int hf \, \mathrm{d}\mu + \int h(1 - f) \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu + \int h(1 - f) \, \mathrm{d}\mu > 1 - \varepsilon - \int h(1 - f) \, \mathrm{d}\mu^- \geqslant 1 - 2\varepsilon - \eta/2$ because $\int h(1 - f) \, \mathrm{d}\mu^- \leqslant \int (1 - f) \, \mathrm{d}\mu^-$.

In case (2) $|\int f \, \mathrm{d}\mu - \int hf \, \mathrm{d}\mu| = |\int_{Pf} (1 - h) f \, \mathrm{d}\mu| \leqslant |\mu|(Pf) < \varepsilon$ and therefore $\int hf \, \mathrm{d}\mu > \int f \, \mathrm{d}\mu - \varepsilon$.

Thus $\int h \, \mathrm{d}\mu > \int f \, \mathrm{d}\mu - \varepsilon + \int h(1 - f) \, \mathrm{d}\mu > 1 - 2\varepsilon - \int h(1 - f) \, \mathrm{d}\mu^- \geqslant 1 - 3\varepsilon - \eta/2$.

(3) If $h(\operatorname{coz} f) = 0$ then $h = h(1 - f)$, and hence $\int h \, \mathrm{d}\mu = \int h(1 - f) \, \mathrm{d}\mu \leqslant \int h(1 - f) \, \mathrm{d}\mu^+ < (\eta + 2\varepsilon)/2$. □

Lemma 3.2. *Let z be a point of a compact space K. Let V be a neighborhood of z and let (λ_α) be a net converging to δ_z with $\|\lambda_\alpha\| \leqslant M$ for each α. Then for each $\varepsilon > 0$ there exists a continuous function f on K, $0 \leqslant f \leqslant 1$ and a subnet (λ_{α_j}) such that $f \equiv 1$ in a neighborhood of z, $\operatorname{supp} f \subset \operatorname{int} V$, $\lambda_{\alpha_j} \in W(\delta_z, \{f, 1 - f\}, \varepsilon)$ and $|\lambda_{\alpha_j}|(Pf) < \varepsilon$ for each j.*

Proof. Let f_1 be a continuous function on K such that $0 \leqslant f_1 \leqslant 1$, $f_1 \equiv 1$ in a neighborhood of z and $\operatorname{supp} f_1 \subset \operatorname{int} V$. There is an index α^1 such that or every $\alpha \geqslant \alpha^1$, $\lambda_\alpha \in W(\delta_z, \{f_1, 1 - f_1\}, \varepsilon)$. We consider two cases:

(1) There exists a subnet $(\lambda_{\alpha_j})_{\alpha_j \geqslant \alpha^1}$ for which the conclusion holds with f_1.

(2) There exists an index $\beta_1 \geqslant \alpha^1$ such that for each $\alpha \geqslant \beta_1$, $|\lambda_\alpha|(Pf_1) \geqslant \varepsilon$. Let n be a positive integer such that $n\varepsilon > M$.

Let f_2 be a continuous function on K such that $0 \leqslant f_2 \leqslant 1$, $f_2 \equiv 1$ in a neighborhood of z and $\operatorname{supp} f_2 \subset \operatorname{int} Z(1 - f_1)$. Let $\alpha^2 \geqslant \beta_1$ be such that for every $\alpha \geqslant \alpha^2$, $\lambda_\alpha \in W(\delta_z, \{f_2, 1 - f_2\}, \varepsilon)$.

Again we consider case (1) if there exists a subnet $(\lambda_{\alpha_j})_{\alpha_j \geqslant \alpha^2}$ for which the conclusion holds for f_2 and case (2) if there exists an index $\beta^2 \geqslant \alpha^2$ such that for each $\alpha \geqslant \beta_2$, $|\lambda_\alpha|(Pf_2) \geqslant \varepsilon$.

We note that case (1) must hold after a finite number of steps. If not, there would be an index $\beta_n \geqslant \alpha^n$ such that for each $\alpha \geqslant \beta_n$, $|\lambda_\alpha|(Pf_n) \geqslant \varepsilon$. Since $\beta_n \geqslant \beta_i$ for all i, we have $|\lambda_{\beta_n}|(Pf_i) \geqslant \varepsilon$ for all $i = 1, \ldots, n$, and these sets are pairwise disjoint, but then $\|\lambda_{\beta_n}\| \geqslant n\varepsilon > M$, contradiction. □

Proof of Theorem 2.1. Suppose on the contrary that $\|T\| = 1 + \eta$ with $0 \leqslant \eta < 1$. Then, by the integral representation of [6, 4.1], there is a continuous function μ mapping $\widehat{X} \times \widehat{Y}$ into $(1 + \eta)B$ whose restriction to S coincides with δ, B being the unit ball of $C(S)^*$ with its weak-star topology and δ the canonical embedding of S into B. We write $\widehat{X} = X \cup \{\infty_X\}$ and $\widehat{Y} = Y \cup \{\infty_Y\}$.

Let $\varepsilon > 0$ such that $\varepsilon < (1 - \eta)/4$. Our purpose is to construct two sequences (t_j) and (z_j) of different points of X and Y, respectively, and also a continuous function h on $\widehat{X} \times \widehat{Y}$ such that the following conditions are satisfied:

$$\int h \, \mathrm{d}\mu(t_j, z_j) > 1 - 2\varepsilon - \eta/2 \quad \text{for all } j,$$

$$\int h \, \mathrm{d}\mu(t_j, z_k) < (\eta + 2\varepsilon)/2 \qquad \text{for all } j > k \geqslant 2.$$

Then if (u, v) is a cluster point in $\widehat{X} \times \widehat{Y}$ of the sequence $\{(t_j, z_j)\}$, by continuity $\mu(u, v)$ is a cluster point of $\{\mu(t_j, z_j)\}$ and so $\int h \, \mathrm{d}\mu(u, v) \geqslant 1 - 2\varepsilon - \eta/2$.

But the point (u, v) is also a cluster point of the set $\{(t_j, z_k) \mid j > k \geqslant 2\}$, so that $\int h \, \mathrm{d}\mu(u, v) \leqslant (\eta + 2\varepsilon)/2$, and this is contradictory with the choice of ε.

In order to construct the desired sequences we will define by induction sequences (x_n), (y_n), (g_n), (h_n), (U_n), (V_n), (A_n) and (B_n) satisfying for each $n \geqslant 1$:

(a) g_n, h_n are continuous functions on S such that $0 \leqslant g_n, h_n \leqslant 1$, $g_n(\infty_X, y_n) = 1$, $g_n(X \times \{\infty_Y\}) = 0$, $h_n(x_n, \infty_Y) = 1$ and $h_n(\{\infty_X\} \times Y) = 0$.

(b) A_n (respectively B_n) is an open neighborhood of ∞_X (respectively ∞_Y) such that $x_{n+1} \in A_n, y_n \in B_n$, $A_{n+1} \subset A_n$, $B_{n+1} \subset B_n$, $\operatorname{supp} g_n \subset \{\infty_X\} \times (B_n \backslash B_{n+1})$ and $\operatorname{supp} h_n \subset (A_{n-1} \backslash A_n) \times \{\infty_Y\}$ (where $A_0 = X$).

(c) U_n (respectively V_n) is an open neighborhood of x_n (respectively y_n) such that

$$U_{n+1} \subset A_n, \quad Z(1 - g_n) \subset V_n, \quad \bigl|\mu(x_{n+1}, y_n)\bigr|(Pg_n) < \varepsilon,$$

$$\mu(A_n \times V_n) \subset W\bigl(\delta_{(\infty_X, y_n)}, \{g_n, 1 - g_n\}, \varepsilon\bigr) \quad \text{and}$$

$$\mu(U_n \times B_n) \subset W\bigl(\delta_{(x_n, \infty_Y)}, \{h_n, 1 - h_n\}, \varepsilon\bigr).$$

We will only see the first step.

Let $x_1 \in X$ and let h_1, be a continuous function on S satisfying (a) for the point (x_1, ∞_Y). By continuity, we have that $\mu^{-1}(W(\delta_{(x_1,\infty_Y)}, \{h_1, 1-h_1\}, \varepsilon))$ is an open neighborhood of (x_1, ∞_Y), and so there exist open neighborhoods U_1 and B_1 of x_1 and ∞_Y respectively, satisfying $\mu(U_1 \times B_1) \subset W(\delta_{(x_1,\infty_Y)}, \{h_1, 1-h_1\}, \varepsilon)$.

Let $y_1 \in B_1$ and let $\{(z_\alpha, y_1)\}$ a net in $X \times Y$ converging to (∞_X, y_1). By continuity, the net $\{\mu(z_\alpha, y_1)\}$ converges to $\delta_{(\infty_X, y_1)}$.

Taking $V = (\{\infty_X\} \times B_1) \setminus \{(\infty_X, \infty_Y)\}$ as a neighborhood of (∞_X, y_1) in Lemma 3.2, there exist a continuous function $0 \leqslant g_1 \leqslant 1$ on S and a subnet $\{\mu(z_{\alpha_j}, y_1)\}$ such that $g_1 \equiv 1$ in a neighborhood of (∞_X, y_1), supp $g_1 \subset V$ and for each j

$$\mu(z_{\alpha_j}, y_1) \in W\big(\delta_{(\infty_X, y_1)}, \{g_1, 1-g_1\}, \varepsilon\big), \quad \big|\mu(z_{\alpha_j}, y_1)\big|(Pg_1) < \varepsilon.$$

If $g_1(\infty_X, s) = 1$ we have that $\delta_{(\infty_X, s)} \in W(\delta_{(\infty_X, y_1)}, \{g_1, 1-g_1\}, \varepsilon)$. Thus by continuity there are open neighborhoods A_s and V_s of ∞_X and s respectively, such that $\mu(A_s \times V_s) \subset W(\delta_{(\infty_X, y_1)}, \{g_1, 1-g_1\}, \varepsilon)$. Now by compactness $Z(1-g_1)$ is covered by a finite number of sets $\{\infty_X\} \times V_{s_1}, \dots, \{\infty_X\} \times V_{s_n}$. Take

$$V_1 = V_{s_1} \cup \dots \cup V_{s_n} \quad \text{and} \quad A_1 = A_{s_1} \cap \dots \cap A_{s_n}.$$

Then A_1 and V_1 are open neighborhoods of ∞_X and y_1 respectively, such that

$$\mu(A_1 \times V_1) \subset W\big(\delta_{(\infty_X, y_1)}, \{g_1, 1-g_1\}, \varepsilon\big)$$

and $Z(1-g_1) \subset V_1$. By taking a smaller A_1 we can guarantee that supp $h_1 \subset (A_0 \setminus A_1) \times \{\infty_Y\}$.

Since $\lim z_{\alpha_j} = \infty_X$, there is an index k such that $z_{\alpha_k} \in A_1$. Take $x_2 = z_{\alpha_k}$ and continue the construction.

We prove now the following

Claim. *Every neighborhood of (∞_X, ∞_Y) in $\infty_X \times \widehat{Y}$ meets infinitely many sets $Z(1-g_n)$.*

On the contrary there is a function $h \in C(S)$ such that $0 \leqslant h \leqslant 1$, $h(X \times \{\infty_Y\}) = 0$ and $h(Z(1-g_n)) = 1$ for all n greater than a certain positive integer. We can suppose that this is true for all positive integer n.

Since $(x_{n+1}, y_n) \in A_n \times V_n$, we have that

$$\mu(x_{n+1}, y_n) \in W\big(\delta_{(\infty_X, y_n)}, \{g_n, 1-g_n\}, \varepsilon\big).$$

Moreover, $h(Z(1-g_n)) = 1$ and $|\mu(x_{n+1}, y_n)|(Pg_n) < \varepsilon$. By Lemma 3.1 we obtain that $\int h \,\mathrm{d}\mu(x_{n+1}, y_n) > 1 - 3\varepsilon - \eta/2$.

Let (u, v) be a cluster point of the sequence $\{(x_{n+1}, y_n)\}$. By continuity $\mu(u, v)$ is a cluster point of $\{\mu(x_{n+1}, y_n)\}$ and so $\int h \,\mathrm{d}\mu(u, v) \geqslant 1 - 3\varepsilon - \eta/2$.

On the other hand, if $k \geqslant j \geqslant 2$, then $(x_j, y_k) \in U_j \times B_j$ and therefore

$$\mu(x_j, y_k) \in W\big(\delta_{(x_j, \infty_Y)}, \{h_j, 1-h_j\}, \varepsilon\big).$$

Since $h(\operatorname{coz} h_j) = 0$, Lemma 3.1 gives $\int h \,\mathrm{d}\mu(x_j, y_k) < (\eta + 2\varepsilon)/2$.

It is easy to check that (u,v) is also a cluster point of $\{(x_j,y_k) \mid k \geqslant j \geqslant 2\}$ and thus, by continuity, $\mu(u,v)$ is a cluster point of the set $\{\mu(x_j,y_k) \mid k \geqslant j \geqslant 2\}$. Therefore $\int h\,\mathrm{d}\mu(u,v) \leqslant (\eta+2\varepsilon)/2$ and this contradicts the choice of ε. The claim is proved.

Since X is pseudocompact and locally compact, by Glicksberg's characterization, there exists a compact subset K of X such that $P_n = \operatorname{int}_X(K \cap U_n) \neq \emptyset$ for infinite positive integers n.

Let n_1 such that $P_{n_1} \neq \emptyset$ and let $t_1 \in P_{n_1}$. Choose a continuous function f_1 on S such that $0 \leqslant f_1 \leqslant 1$, $\operatorname{supp} f_1 \subset P_{n_1} \times \{\infty_Y\}$ and $f_1(t_1,\infty_Y) = 1$. By continuity, there is a neighborhood G_1 of ∞_Y such that $\mu(\{t_1\} \times G_1) \subset W(\delta_{(t_1,\infty_Y)}, \{f_1, 1-f_1\}, \varepsilon)$. By the claim, G_1 meets an infinite number of sets $Z(1-g_n)$. Thus there exists a positive integer $m_1 \geqslant n_1$ and a point $z_1 \in Z(1-g_{m_1}) \cap G_1$. Let $n_2 > m_1$ be such that $P_{n_2} \neq \emptyset$ and take a point $t_2 \in P_{n_2}$. Continuing by induction, we obtain the sequences (t_j), (f_j), (z_j), (P_{n_j}) and (G_j) such that for $j \geqslant 1$

$$t_j \in P_{n_j}, \quad z_j \in Z(1-g_{m_j}) \cap G_j, \quad n_j \leqslant m_j < n_{j+1} \quad \text{and}$$
$$\mu(\{t_j\} \times G_j) \subset W(\delta_{(t_j,\infty_Y)}, \{f_j, 1-f_j\}, \varepsilon).$$

Let $h \in C(S)$ such that $0 \leqslant h \leqslant 1$, $h(K \times \{\infty_Y\}) = 1$ and $h(\{\infty_X\} \times Y) = 0$.
We will see that (t_j), (z_j) and h satisfy the conditions stated at the begining.
Since $z_j \in G_j$, we have

$$\mu(t_j,z_j) \in W(\delta_{(t_j,\infty_Y)}, \{f_j, 1-f_j\}, \varepsilon).$$

Moreover, $h(\operatorname{coz} f_j) = 1$ and so by Lemma 3.1 we conclude that $\int h\,\mathrm{d}\mu(t_j,z_j) > 1 - 2\varepsilon - \eta/2$ for all j.

On the other hand, by part (c) $Z(1-g_i) \subset V_i$ and $U_{i+1} \subset A_i$ for all i, and consequently we have that $(t_j,z_k) \in P_{n_j} \times V_{m_k}$ for all $j > k \geqslant 2$. But also $n_k \leqslant m_k < n_j$ and then $U_{n_j} \subset A_{n_j-1} \subset A_{m_k}$. Thus $(t_j,z_k) \in A_{m_k} \times V_{m_k}$ and therefore

$$\mu(t_j,z_k) \in W(\delta_{(\infty_X,y_{m_k})}, \{g_{m_k}, 1-g_{m_k}\}, \varepsilon).$$

Since $h(\operatorname{coz} g_{m_k}) = 0$, we conclude by Lemma 3.1 that $\int h\,\mathrm{d}\mu(t_j,z_k) < (\eta+2\varepsilon)/2$. This concludes the proof. □

4. Remarks

The hypothesis about pseudocompactness in the above theorem can not be dropped. For instance, if $\mathbb{N}$ is the discrete space of positive integers, then $\widehat{\mathbb{N}} \times \widehat{\mathbb{N}}$ is a compact metric space and by the Borsuk–Dugundji theorem there exists a norm one linear extension operator from $C((\widehat{\mathbb{N}} \times \widehat{\mathbb{N}}) \backslash (\mathbb{N} \times \mathbb{N}))$ to $C(\widehat{\mathbb{N}} \times \widehat{\mathbb{N}})$.

The following example shows that in certain cases the lower bounds obtained in Theorems 2.1 and 2.5 for the norm of linear extension operators are the best possible.

Example. We write ω_1 for the first uncountable ordinal. $[1,\omega_1)$ (respectively $[1,\omega_1]$) is the set of all ordinals α such that $1 \leqslant \alpha < \omega_1$ (respectively $1 \leqslant \alpha \leqslant \omega_1$). We consider in $[1,\omega_1)$ and $[1,\omega_1]$ the order topology. Thus we know that $[1,\omega_1)$ is a sequentially compact (i.e., pseudocompact) space and that $[1,\omega_1]$ is the Alexandroff compactification of $[1,\omega_1)$ [4, Example 3.6.10].

Let X and Y be two copies of $[1,\omega_1)$ and let $S = (\widehat{X} \times \widehat{Y}) \backslash (X \times Y)$. For each $f \in C(S)$ define a real function $T(f)$ on $\widehat{X} \times \widehat{Y}$ by

$$T(f)(x,y) = \begin{cases} \big(f(\omega_1,y) + f(y,\omega_1)\big)/2 & \text{if } y \leqslant \omega_1, \\ f(\omega_1,y) + \big(f(x,\omega_1) - f(\omega_1,x)\big)/2 & \text{if } y < x \leqslant \omega_1, \\ f(x,\omega_1) + \big(f(\omega_1,y) - f(y,\omega_1)\big)/2 & \text{if } x < y < \omega_1. \end{cases}$$

It is easy to see that the restriction of $T(f)$ to the closed subsets $\{(x,y) \mid x \leqslant y\}$ and $\{(x,y) \mid y \leqslant x\}$ is continuous. Therefore $T(f)$ is continuous on $\widehat{X} \times \widehat{Y}$. Clearly T is a linear extension operator from $C(S)$ into $C(\widehat{X} \times \widehat{Y})$ such that $\|T\| \leqslant 2$. By Theorem 2.1 the norm must be equal to 2.

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